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Linear Estimation of the Number of Zeros of Abelian Integrals for Some Cubic Isochronous Centers

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This paper consists of two parts. In the first part we study the relationship between conic centers (all orbits near a singular point of center type are conics) and isochronous centers of polynomial systems. In the second part we study the number of limit cycles that bifurcate from the periodic orbits of cubic reversible isochronous centers having all their orbits formed by conics, when we perturb such systems inside the class of all polynomial systems of degree n.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The main open problem in the qualitative theory of real planar differential systems is the determination of limit cycles. A classical way to obtain limit cycles is perturbing the periodic orbits of a center. There are several methods for studying the bifurcated limit cycles from a center. The major part of the methods is based either on the Poincaré return map or on the Poincaré–Melnikov integral or Abelian integral which are equivalent in the plane (see for instance [1]). Recently some other methods were presented, based on the inverse integrating factor (see [7]); others have been based on the reduction of the problem to a one dimensional differential equation (see [13] and [16]). In general these methods are difficult to apply for studying the limit cycles that bifurcate from the periodic orbits of a center when the

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system is integrable but not Hamiltonian. As far as we know few papers study the non-Hamiltonian centers; see for instance [2, 4, 8, 11, 13, 14, 16].

By definition a *polynomial system* is a differential system of the form

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y), \tag{1}$$

where P and Q are polynomials with real coefficients. We say that $n = \max\{\deg P, \deg Q\}$ is the *degree* of the polynomial system.

In what follows for $n=2$ or $n=3$ systems (1) are called *quadratic* or *cubic* systems, respectively. Many authors have studied the limit cycles which bifurcate from periodic orbits of a center for a quadratic system; see, for instance, [4, 9, 15, 14, 19, 22–24, 26].

In this paper we will study the number of zeros of Abelian integrals for some cubic isochronous centers, which are characterized in the following proposition.

PROPOSITION 1. *After a linear change of variables and a rescaling of the time variable (if necessary) a polynomial system* (1) *satisfying*

(i) *the polynomials P and Q are coprime inside the ring of real polynomials in two variables,*

(ii) *it has a center at the origin* $(0, 0)$, *and*

(iii) *all its orbits near* $(0, 0)$ *are conics*

goes over to one of the systems in Table I, where $(a, b) \in \mathbf{R}^2 \setminus \{(0, \pm 1)\}$. *Moreover all these polynomial systems are isochronous.*

Proposition 1 will be proved in Section 2.

TABLE I

Name	System	Integrating factor	First integral
S_0	$\dot{x} = -y$ $\dot{y} = x$	1	x^2+y^2
S_1	$\dot{x} = -y+\frac{1}{2}x^2-\frac{1}{2}y^2$ $\dot{y} = x(1+y)$	$\dfrac{1}{(1+y)^2}$	$\dfrac{x^2+y^2}{1+y}$
S_2	$\dot{x} = -y+x^2$ $\dot{y} = x(1+y)$	$\dfrac{1}{(1+y)^3}$	$\dfrac{x^2+y^2}{(1+y)^2}$
S^*	$\dot{x} = -y+bx^2-2axy-by^2+x^2y$ $\dot{y} = x+ax^2+2bxy-ay^2+xy^2$	$\dfrac{1}{(1+2ax+2by+y^2)^2}$	$\dfrac{x^2+y^2}{1+2ax+2by+y^2}$

We remark that multiplying $\dot{x}$ and $\dot{y}$ in the systems of Proposition 1 by an arbitrary polynomial we get, up to affine transformations of the coordinates, all polynomial systems with a center such that all their orbits are conics. We also note that Zoladek in [26] and Mardesic *et al.* in [20] mention implicitly (at least for quadratic systems) some relationship between the fact that all orbits of a center are conics and the fact that the center is isochronous. Also they mentioned that there may be some relationship between the isochronous centers and more regular first integrals.

We note that the perturbation of the linear center S_0 of Proposition 1 inside the class of all polynomial systems of degree n has been studied by several authors; see for instance [1] and [6]. There are four quadratic isochronous centers S_i, $1 \leqslant i \leqslant 4$, by using the notation of paper [20]; two of them are presented in Proposition 1 and the other two can be transformed to S_1 and S_2, respectively, but via nonlinear transformation of coordinates. The perturbation of the quadratic isochronous centers inside the class of all polynomials systems of degree n has been studied in [14]; the case $n = 2$ was studied by Chicone and Jacobs [2]. In [14] we obtain the exact upper bounds of the number of zeros of Abelian integrals for three of these centers and an upper bound for the remaining one; all the bounds depend linearly with the degree of perturbations.

The main result of the second part of this paper is to study the number of the limit cycles that bifurcate from the periodic orbits of cubic reversible isochronous centers of Proposition 1 when we perturb such systems inside the class of all polynomial systems of degree n. The phase portraits of systems S^* are given in Fig. 1. Here we say that a center of a polynomial system (1) is *reversible* if there exists a straight line through the center such that the solutions of the system are symmetric with respect to this straight line and a reversibility of the time variable.

Obviously, system S^* is reversible if and only if $ab = 0$. There are the following three different phase portraits for reversible systems inside the class S^*:

(A) $a = 0$, $0 < |b| < 1$;

(B) $a = 0$, $|b| > 1$ or $b = 0$, $a \neq 0$,

(C) $a = b = 0$.

The technique used in this paper for studying the perturbation of integrable but non-Hamiltonian systems is classical. It consists of writing a non-Hamiltonian center in a Hamiltonian one multiplying by a suitable integrating factor. The key point in our approach is that, by Green's theorem, we will compute the Abelian integral through a double integral. These double integrals for these cubic isochronous centers are very easy to

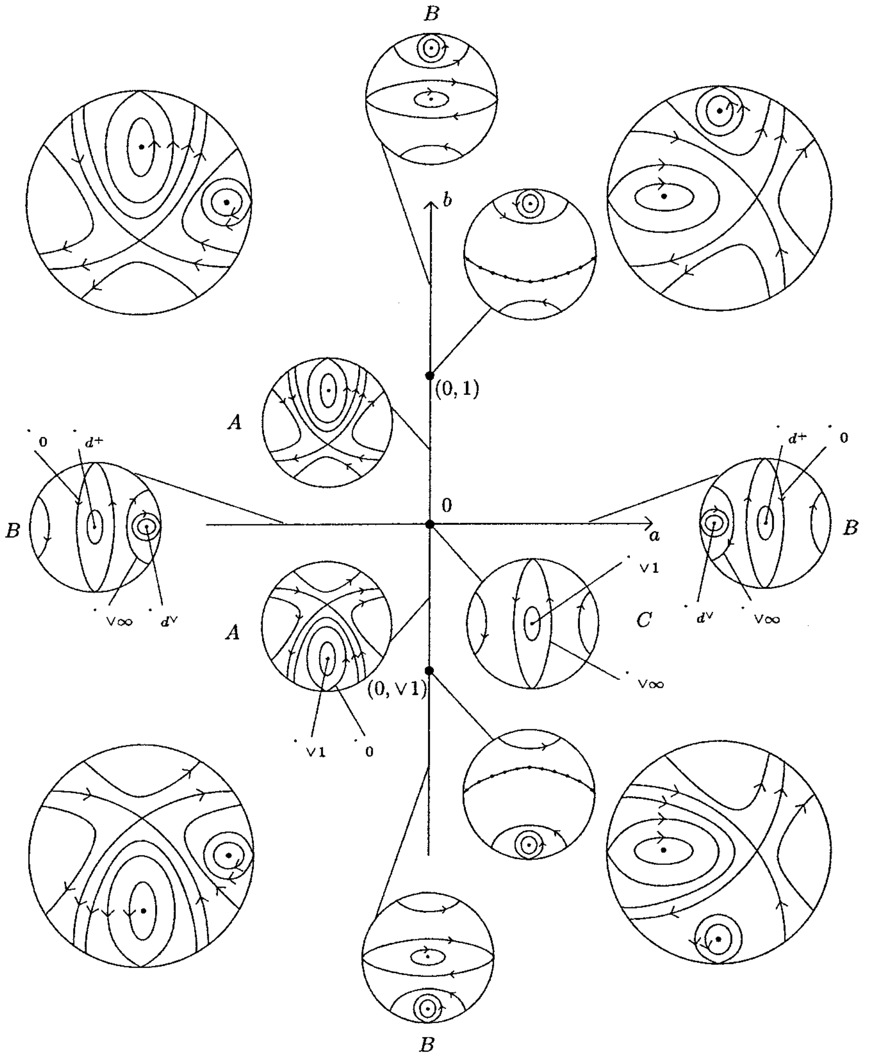

FIG. 1. The phase portraits of the isochronous centers of system S^*.

compute in comparison with the usual single Abelian integral. This technique was used by first time in [14]. Thus, our main result is the following one.

THEOREM 2. *When we perturb the three cubic reversible isochronous systems* (A), (B), *and* (C) *inside the class of all polynomial systems of degree* n *an upper bound for the number of zeros (taking into account their multiplicity) of the Abelian integral associated to system:*

(A) *is 2 for $n \leqslant 2$; 5 for $n=3$; 6 for $n=4$; and $(3n+1)/2$ if $n \geqslant 5$ odd, and $(3n/2)-1$ if $n \geqslant 6$ even;*

(B) *is 1 for $n=0, 1, 2$; 4 for $n=3$; 5 for $n=4$; for $n \geqslant 5$ is $3(n-1)/2$ if n is odd and $(3n/2)-1$ if n is even;*

(C) *is 0 for $n=0$; 1 for $n=1, 2$; and $2[(n+1)/2]$ for $n \geqslant 3$.*

Statements (A), (B), and (C) of Theorem 2 are proved in Sections 3, 4, and 5, respectively.

We note that in spite of our systems being non-Hamiltonian the upper bounds for the number of zeros of the Abelian integrals given in Theorem 2 are all linear in the degree n of the polynomial perturbation, which is in accord with the traditional prediction and some known results for the perturbation of Hamiltonian centers; see for instance [5, 10, 15–17, 21–23, 25].

2. PROOF OF PROPOSITION

We start this section proving a preliminary result that we shall use in the proof of Proposition 1.

LEMMA 3. *Let $f_1 = x^2+y^2-1$ and $f_2 = f_2(x, y)$ be a polynomial of degree 2 with real coefficients; then the system*

$$f_1 = 0, \qquad f_2 = 0,$$

has no solutions in $\mathbf{C}^2$ if and only if there exist constants $C_1, C_2 \in \mathbf{R}$ with $C_2 \neq 0$ such that $f_2 = C_1 f_1 + C_2$.

Proof. Consider the polynomials f_1, f_2 as the elements of $\mathbf{C}[x][y]$,

$$f_1 = y^2+x^2-1, \qquad f_2 = a_0 y^2 + p(x)\, y + q(x),$$

where p, q are polynomials of degree at most 1 and 2, respectively, and a_0 is equal to either 0 or 1. The resultant of f_1, f_2 is

$$\begin{aligned} R_y(f_1, f_2) = {} & (a^2+d^2)\, x^4 + (2ab+2de)\, x^3 + (b^2+2ac+e^2-d^2)\, x^2 \\ & + (2bc-2de)\, x + c^2 - e^2, \end{aligned}$$

where the constants a, b, c, d, e are defined as follows:

$$dx+e = p(x), \qquad ax^2+bx+c = \begin{cases} q(x), & \text{if} \quad a_0 = 0; \\ q(x)-x^2+1, & \text{if} \quad a_0 = 1. \end{cases}$$

If the system $f_1=0$, $f_2=0$ has no solutions, then its resultant must be a nonzero constant. This is equivalent to

$$a=b=d=e=0, \qquad c\neq 0,$$

which implies the conclusion of the lemma. ∎

Now we prove Proposition 1. Assume that all the orbits of polynomial system (1) near the center are conics and that P and Q are coprime. We claim that system (1) has a rational first integral H_1/H_2, where $\max\{\deg H_1, \deg H_2\}=2$. Indeed, by Jouanolou's results [12] (see [3] for an easier proof for the planar polynomial systems), if a polynomial system of degree m admits $q>2+[m(m+1)/2]$ algebraic phase curves $f_i=0$, $i=1, ..., q$, being $f_1, ..., f_q$ relatively prime in $\mathbf{C}[x, y]$, then there exist integers n_i not all zero such that $f_1^{n_1}\cdots f_q^{n_q}$ is a rational first integral. Denote by $f=\prod_{n_i\geqslant 0} f_i^{n_i}$, $g=\prod_{n_i<0} f_i^{n_i}$. Since system (1) has a center and all phase curves near it are conic, we may assume, without loss of generality, that f is not a constant and has a divisor of the form x^2+y^2-1, and there exists an open interval $I\subset\mathbf{R}$ such that for any constant $c\in I$, $f/g=c$ determines an ellipse or circle E_c: $h_c(x, y)=0$, where h_c depends continuously on c, and it is a polynomial of degree 2 with real coefficients. This implies that the polynomial $f-cg\in\mathbf{C}[x, y]$ has the divisor h_c; i.e.,

$$f(x, y)-cg(x, y)=h_c(x, y)\,k_c(x, y),$$

for some polynomial k_c.

Case 1. There exists a subinterval $J\subset I$ such that the system of equations

$$x^2+y^2-1=0, \qquad h_c=0$$

has no solutions in $\mathbf{C}^2$ for any $c\in J$. By Lemma 3, $h_c=C_1(c)(x^2+y^2)+C_2(c)$. This implies that all phase curves E_c are concentric circles. Hence, x^2+y^2 is a first integral of system (1).

Case 2. There exists a nonzero constant $\bar{c}\in I$ such that the system of equations

$$x^2+y^2-1=0, \qquad h_{\bar{c}}=0$$

has solutions in $\mathbf{C}^2$. Thus, the set

$$A=\{(x, y)\in\mathbf{C}^2 : x^2+y^2-1=0,\, g(x, y)=0\}$$

is not empty. Note that the set A is finite and

$$B_c = \{(x, y) \in \mathbf{C}^2 : x^2 + y^2 - 1 = 0, h_c = 0\} \subset A, \qquad \text{for} \quad c \neq 0;$$

we have

$$B_c = B_{\bar{c}} \qquad \text{for} \quad |c - \bar{c}| \ll 1,$$

which implies $h_c = C_1(x^2 + y^2 - 1) + C_2 h_{\bar{c}}$ for some constants $C_i = C_i(c)$. Therefore, $(x^2 + y^2 - 1)/h_{\bar{c}}$ is a first integral of system (1), and the claim is proved.

Now, we have proved that system (1) has a rational first integral $H = f/g$ with $\max\{\deg f, \deg g\} = 2$. Without loss of generality, we may assume that $H = 0$ corresponds to the center $(0, 0)$, which implies that $g(0, 0) \neq 0$, $f(0, 0) = 0$. Therefore, after a rotation of coordinates and a rescaling of x and y, f can be transformed into the form $f(x, y) = x^2 + y^2$, and the first integral can be written into the form

$$H(x, y) = \frac{x^2 + y^2}{1 + ax + by + cxy + dx^2 + ey^2}.$$

Taking another rotation of coordinates, we can eliminate the term xy in the denominator. Obviously, the function

$$\frac{H}{1 - dH} = \frac{x^2 + y^2}{1 + ax + by + (e - d)\, y^2}$$

is also a first integral. In short, we get a first integral of system (1) as follows:

$$H(x, y) = \frac{f}{g}, \qquad f = x^2 + y^2, \qquad g = 1 + ax + by + cy^2. \tag{2}$$

Obviously, the function H in (2) is a first integral of the following cubic system

$$\begin{aligned} \dot{x} &= g'_y f - f'_y g, \\ \dot{y} &= f'_x g - g'_x f. \end{aligned} \tag{3}$$

By removing the common factor in $\dot{x}$ and $\dot{y}$ (if they have), we get a polynomial system of degree $\leqslant 3$ as follows

$$\begin{aligned} \dot{x} &= \tilde{P}(x, y), \\ \dot{y} &= \tilde{Q}(x, y), \end{aligned} \tag{4}$$

where $\tilde{P}$ and $\tilde{Q}$ are coprime. Since (1) and (4) have a common first integral H, we obtain

$$\frac{P}{Q} \equiv \frac{\tilde{P}}{\tilde{Q}},$$

which, by the fact that $(P, Q) = 1$ and $(\tilde{P}, \tilde{Q}) = 1$, implies

$$\frac{P}{\tilde{P}} = \frac{Q}{\tilde{Q}} = \text{constant}.$$

Next we prove that system (4) can be written into the forms given in Proposition 1.

(i) if $g \equiv 1$, then system (4) is the linear center S_0.

(ii) if $c = 0$, and $a^2 + b^2 > 0$, then after a rotation of coordinates, $g = 1 + y$. Therefore system (4) is the system S_1.

(iii) if $a = 0$ and $b^2 = 4c \neq 0$, then after a rescaling of y and x, $g = (1 + y)^2$. Hence system (4) is the system S_2.

(iv) if $c \neq 0$ and $a^2 + (b^2 - 4c)^2 > 0$, then rescaling x and y, we can get $c = \pm 1$ and system (4) becomes

$$\begin{aligned} \dot{x} &= -y + bx^2 - 2axy - by^2 + cx^2 y, \\ \dot{y} &= x + ax^2 + 2bxy - ay^2 + cxy^2, \qquad c = \pm 1. \end{aligned}$$

If $c = -1$, we can change the sign of c using the transformation $(x, y, t) \to (y, x, -t)$. Thus we get the system S^*.

Now we prove that all centers given in Proposition 1 are isochronous. The isochronicity of systems S_0, S_1, and S_2 is well known (see [18]); here we only prove it for system S^*.

System S^* has the first integral

$$H(x, y) = \frac{x^2 + y^2}{1 + 2ax + 2by + y^2}.$$

Let

$$u = \frac{x}{\sqrt{1 + 2ax + 2by + y^2}}, \qquad v = \frac{y}{\sqrt{1 + 2ax + 2by + y^2}}.$$

Then

$$x = -\frac{(au + bv + \sqrt{\Delta})\, u}{(v^2 - 1)}, \qquad y = -\frac{(au + bv + \sqrt{\Delta})\, v}{(v^2 - 1)},$$

where $\Delta = (au+bv)^2 - v^2 + 1$, and system S^* goes over to

$$\dot{u} = -v(1+ax+by)\,,$$
$$\dot{v} = u(1+ax+by)\,.$$

All periodic orbits of this system are concentric circles. Next we prove that they have the same period.

Let $u = r\cos\theta$ and $v = r\sin\theta$. Then

$$\begin{aligned}\frac{d\theta}{dt} &= 1+ax+by\\ &= 1-\left(\frac{au+bv+\sqrt{\Delta}}{(v^2-1)}\right)(au+bv)\\ &= -\frac{\Delta+\sqrt{\Delta}\,(au+bv)}{(v^2-1)}\\ &= \frac{\sqrt{\Delta}}{\sqrt{\Delta}-(au+bv)}.\end{aligned}$$

Then

$$\begin{aligned}T = \int_0^T dt &= \int_0^{2\pi} \frac{\sqrt{\Delta}-(au+bv)}{\sqrt{\Delta}}\,d\theta\\ &= 2\pi - \int_0^{2\pi} \frac{a\cos\theta + b\sin\theta}{\sqrt{(a\cos\theta+b\sin\theta)^2 - \sin^2\theta + r^{-2}}}\,d\theta\\ &= 2\pi.\end{aligned}$$

Thus, the center at the origin is an isochronous one. We claim that the second center, which exists for $(a, b) \in \mathbf{R}^2 \setminus \{a=0, |b| \leqslant 1\}$, is also isochronous. Indeed, after moving the origin to the second center, all previous arguments work.

The proof of Proposition 1 is completed.

3. LINEAR ESTIMATION OF THE NUMBER OF ZEROS OF ABELIAN INTEGRAL FOR SYSTEM A

Consider the polynomial perturbation of system A,

$$\begin{aligned}\dot{x} &= -y + bx^2 - by^2 + x^2y + \varepsilon P(x, y),\\ \dot{y} &= x + 2bxy + xy^2 + \varepsilon Q(x, y),\end{aligned} \tag{5}$$

where $0<|b|<1$, P and Q are polynomials of degree n. For $\varepsilon=0$, system (5) has a first integral $H=(x^2-1-2by)/(1+2by+y^2)$. For $-1<h<0$, the level curves Γ_h: $H=h$ are periodic orbits which surround the center Γ_{-1}: $(0, 0)$ (see Fig. 1). Multiplying by the integrating factor $2/(1+2by+y^2)^2$, system (5) is changed to the form

$$\dot{x}=-\frac{\partial H}{\partial y}+\varepsilon\frac{2P}{(1+2by+y^2)^2},$$
$$\dot{y}=\frac{\partial H}{\partial x}+\varepsilon\frac{2Q}{(1+2by+y^2)^2}.$$

The Abelian integral for the above system is defined as

$$M(h)=\oint_{H=h}\left(\frac{2P}{(1+2by+y^2)^2}\,dy-\frac{2Q}{(1+2by+y^2)^2}\,dx\right), \qquad -1\leqslant h<0.$$

By Green's formula

$$\begin{aligned}M(h)&=2\iint_{H\leqslant h}\left[\frac{\partial}{\partial x}\left(\frac{P}{(1+2by+y^2)^2}\right)+\frac{\partial}{\partial y}\left(\frac{Q}{(1+2by+y^2)^2}\right)\right]dx\,dy\\&=2\iint_{H\leqslant h}\left[\frac{P'_x+Q'_y}{(1+2by+y^2)^2}-\frac{4(b+y)\,Q}{(1+2by+y^2)^3}\right]dx\,dy.\end{aligned}$$

Set $x=u$ and $y=\sqrt{1-b^2}\,v-b$. Then

$$H(u,v)=H(u,\sqrt{1-b^2}\,v-b)=\frac{u^2-1-2b\sqrt{1-b^2}\,v+2b^2}{(1-b^2)(1+v^2)},$$

and

$$M(h)=2\iint_{H\leqslant h}\left[\frac{P'_x+Q'_y}{(1-b^2)^2\,(1+v^2)^2}-\frac{4(b+y)\,Q}{(1-b^2)^3\,(1+v^2)^3}\right]\sqrt{1-b^2}\,du\,dv.$$

Let

$$\frac{2P'_x+2Q'_y}{(1-b^2)}=\sum_{i+j\leqslant n-1}(1-b^2)^{-\frac{i}{2}}\,a_{ij}u^iv^j,$$
$$\frac{8(b+y)\,Q}{(1-b^2)^2}=\sum_{i+j\leqslant n}(1-b^2)^{-\frac{i}{2}}\,b_{ij}u^iv^{j+1}.$$

Then

$$
\begin{aligned}
M(h) &= \sum_{i+j \leqslant n-1} a_{ij}(1-b^2)^{-\frac{i}{2}-\frac{1}{2}} \iint_{H \leqslant h} \frac{u^i v^j}{(1+v^2)^2}\, du\, dv \\
&\quad - \sum_{i+j \leqslant n} b_{ij}(1-b^2)^{-\frac{i}{2}-\frac{1}{2}} \iint_{H \leqslant h} \frac{u^i v^{j+1}}{(1+v^2)^3}\, du\, dv \\
&= \sum_{2i+j \leqslant n-1} a_{2ij}(1-b^2)^{-i-\frac{1}{2}} \iint_{H \leqslant h} \frac{u^{2i} v^j}{(1+v^2)^3}\, du\, dv \\
&\quad - \sum_{2i+j \leqslant n} b_{2ij}(1-b^2)^{-i-\frac{1}{2}} \iint_{H \leqslant h} \frac{u^{2i} v^{j+1}}{(1+v^2)^3}\, du\, dv \\
&= \sum_{2i+j \leqslant n-1} \frac{2a_{2ij}}{(2i+1)} \int_{v_1}^{v_2} \frac{v^j \Delta^i \sqrt{\Delta}}{(1+v^2)^2}\, dv \\
&\quad - \sum_{2i+j \leqslant n} \frac{2b_{2ij}}{(2i+1)} \int_{v_1}^{v_2} \frac{v^{j+1} \Delta^i \sqrt{\Delta}}{(1+v^2)^3}\, dv,
\end{aligned}
$$

where $\Delta = h(1+v^2) + 2bv/\sqrt{1-b^2} + (1-2b^2)/(1-b^2)$, v_1, v_2 are two roots of the equation $\Delta = 0$, such that

$$
\begin{aligned}
v_1 + v_2 &= \frac{\bar{b}}{h}, & v_1 v_2 &= 1 + \frac{\bar{a}}{h}, \\
\bar{b} &= -\frac{2b}{\sqrt{1-b^2}}, & \bar{a} &= \frac{1-2b^2}{1-b^2}.
\end{aligned} \tag{6}
$$

If

$$
\begin{aligned}
&\sum_{2i+j \leqslant n-1} \frac{2a_{2ij}}{2i+1} (1+v^2)\, v^j \Delta^i - \sum_{2i+j \leqslant n} \frac{2b_{2ij}}{(2i+1)}\, v^{j+1} \Delta^i \\
&\quad = \sum_{k=0}^{[\frac{n}{2}]} m_k(h)\, v(1+v^2)^k + \sum_{k=0}^{[\frac{n+1}{2}]} \bar{m}_k(h)(1+v^2)^k,
\end{aligned}
$$

where m_k, $\bar{m}_k$ are polynomials of h with

$$
\deg m_k \leqslant k\,, \qquad \deg \bar{m}_k \leqslant k\,, \tag{7}
$$

then

$$
M(h) = \sum_{k=0}^{[\frac{n}{2}]} m_k(h)\, J_{k-3} + \sum_{k=0}^{[\frac{n+1}{2}]} \bar{m}_k(h)\, I_{k-3}, \tag{8}
$$

where

$$I_k = \sqrt{-h}\int_{v_1}^{v_2} (1+v^2)^k \sqrt{(v-v_1)(v_2-v)}\, dv,$$

$$J_k = \sqrt{-h}\int_{v_1}^{v_2} v(1+v^2)^k \sqrt{(v-v_1)(v_2-v)}\, dv.$$

If $\sqrt{(v_2-v)(v-v_1)} = t(v-v_1)$, then

$$I_k = 2\sqrt{-h}\,(v_2-v_1)^2 \int_0^\infty \frac{t^2[(1+t^2)^2+(v_2+v_1t^2)^2]^k}{(1+t^2)^{2k+3}}\, dt,$$

$$J_k = 2\sqrt{-h}\,(v_2-v_1)^2 \int_0^\infty \frac{t^2(v_2+v_1t^2)[(1+t^2)^2+(v_2+v_1t^2)^2]^k}{(1+t^2)^{2k+4}}\, dt.$$

Consequently we have that

$$\begin{aligned} I_k(v_1, v_2) &= 2\sqrt{-h}\,(v_2-v_1)^2 \int_0^\infty \frac{t^2[(1+t^2)^2+(v_2+v_1t^2)^2]^k}{(1+t^2)^{2k+3}}\, dt \qquad (t=u^{-1}) \\ &= 2\sqrt{-h}\,(v_2-v_1)^2 \int_0^\infty \frac{u^2[(1+u^2)^2+(v_1+v_2u^2)^2]^k}{(1+u^2)^{2k+3}}\, du \\ &= I_k(v_2, v_1). \end{aligned}$$

Similarly, we have $J_k(v_1, v_2) = J_k(v_2, v_1)$. These mean that for $k \geqslant 0$, I_k and J_k are symmetric polynomials of (v_1, v_2) of degree $2k+2$ and $2k+3$, respectively. Therefore, there exist constants c_{ij}, d_{ij} such that

$$\begin{aligned} I_k &= 2\sqrt{-h}\,(v_2-v_1)^2 \sum_{i+2j\leqslant 2k} c_{ij}(v_1+v_2)^i\,(v_1v_2)^j \qquad \text{(by (6))} \\ &= 2\sqrt{-h}\,(v_2-v_1)^2 \sum_{i+2j\leqslant 2k} c_{ij}\,\frac{\bar{b}^i}{h^i}\left(1+\frac{\bar{a}}{h}\right)^j \\ &= 2\sqrt{-h}\,(v_2-v_1)^2\, h^{-2k} P_{2k}(h), \end{aligned} \tag{9}$$

and

$$\begin{aligned} J_k &= 2\sqrt{-h}\,(v_2-v_1)^2 \sum_{i+2j\leqslant 2k+1} d_{ij}(v_1+v_2)^i\,(v_1v_2)^j \\ &= 2\sqrt{-h}\,(v_2-v_1)^2 \sum_{i+2j\leqslant 2k+1} d_{ij}\,\frac{\bar{b}^i}{h^i}\left(1+\frac{\bar{a}}{h}\right)^j \\ &= 2\sqrt{-h}\,(v_2-v_1)^2\, h^{-2k-1} P_{2k+1}(h), \end{aligned} \tag{10}$$

where P_k denotes a polynomial of degree k.

Next we calculate I_k, J_k for $k<0$. Note that

$$(1+t^2)^2+(v_2+v_1t^2)^2=(1+v_1^2)(t^2+\alpha t+\beta)(t^2-\alpha t+\beta),$$

$$\alpha=\sqrt{\frac{2(\sqrt{(1+v_1^2)(1+v_2^2)}-1-v_1v_2)}{(1+v_1^2)}},\qquad \beta=\sqrt{\frac{1+v_2^2}{1+v_1^2}},\tag{11}$$

we obtain

$$I_k=2\sqrt{-h}\,(v_2-v_1)^2\,(1+v_1^2)^k\,U_k,\tag{12}$$

$$J_k=2\sqrt{-h}\,(v_2-v_1)^2\,(1+v_1^2)^k\,V_k,\tag{13}$$

where

$$U_k=\int_0^\infty\frac{t^2(t^2+\alpha t+\beta)^k\,(t^2-\alpha t+\beta)^k}{(1+t^2)^{2k+3}}\,dt,$$

$$V_k=\int_0^\infty\frac{t^2(v_2+v_1t^2)(t^2+\alpha t+\beta)^k\,(t^2-\alpha t+\beta)^k}{(1+t^2)^{2k+4}}\,dt.$$

Computing we get

$$U_{-1}=-\frac{\pi}{2}\left[\frac{\sqrt{\Delta}-\beta-1}{(1-2\beta+\beta^2+\alpha^2)\sqrt{\Delta}}\right],$$

$$U_{-2}=\frac{\pi}{4}\,\frac{\beta+1}{\beta\Delta^{\frac{3}{2}}},$$

$$U_{-3}=-\frac{\pi}{16}\,\frac{-9+\alpha^2-7\beta^{-2}-7\beta+\beta^{-3}\alpha^2-9\beta^{-1}}{\Delta^{\frac{5}{2}}},$$

and

$$V_{-1}=\frac{\pi}{(\beta^2-2\beta+1+\alpha^2)^2\sqrt{\Delta}}$$

$$\Bigg[\left(-\frac{3}{4}\beta^2\sqrt{\Delta}-\frac{3}{2}\beta+\frac{1}{2}\beta\sqrt{\Delta}+\frac{1}{2}\alpha^2+\beta^2+\frac{1}{2}\beta^3+\frac{1}{4}\sqrt{\Delta}-\frac{1}{4}\alpha^2\sqrt{\Delta}\right)v_1$$

$$+\left(\frac{1}{4}\beta^2\sqrt{\Delta}+\frac{1}{2}\alpha^2\beta+\beta+\frac{1}{2}\beta\sqrt{\Delta}+\frac{1}{2}-\frac{3}{4}\sqrt{\Delta}-\frac{3}{2}\beta^2-\frac{1}{4}\alpha^2\sqrt{\Delta}\right)v_2\Bigg],$$

$$V_{-2}=\frac{\pi}{4}\,\frac{(v_1+\beta^{-1}v_2)}{\Delta^{\frac{3}{2}}},$$

$$V_{-3}=\frac{\pi}{16\Delta^{\frac{5}{2}}}\,[(7\beta+6-\alpha^2+3\beta^{-1})\,v_1+(3+6\beta^{-1}-\beta^{-3}\alpha^2+7\beta^{-2})\,v_2],$$

where $\varDelta = 4\beta - \alpha^2$. Substituting formulas (6) and (11) into U_{-i} and V_{-i}, and using (12) and (13), we get

$$\begin{aligned} I_{-1} &= \frac{\pi(h+1))}{1+\sqrt{-h}}, \\ I_{-2} &= (h+1)\, P_0(h), \\ I_{-3} &= (h+1)\, P_1(h), \end{aligned}$$

and

$$\begin{aligned} J_{-1} &= \frac{(h+1)}{\sqrt{-h}\,(\sqrt{-h}+1)}\, \bar{P}_0(h), \\ J_{-2} &= (h+1)\, \tilde{P}_0(h), \\ J_{-3} &= (h+1)\, P_2(h), \end{aligned} \tag{14}$$

where from now on P_k, $\bar{P}_k$, and $\tilde{P}_k$ denote polynomials of degree k.

From (7), (8), and (14), we obtain

$$M(h) = \sum_{k=0}^{1} m_k(h)\, J_{k-3} + \sum_{k=0}^{1} \bar{m}_k(h)\, I_{k-3} = (h+1)\, P_2(h), \qquad \text{for} \quad n \leqslant 2, \tag{15}$$

$$\begin{aligned} M(h) &= \sum_{k=0}^{1} m_k(h)\, J_{k-3} + \sum_{k=0}^{2} \bar{m}_k(h)\, I_{k-3} \\ &= \frac{(h+1)}{P_1(\sqrt{-h})}\,(P_1(\sqrt{-h})\, P_2(h) + \bar{P}_2(h)), \qquad \text{for} \quad n = 3, \end{aligned} \tag{16}$$

$$\begin{aligned} M(h) &= \sum_{k=0}^{2} m_k(h)\, J_{k-3} + \sum_{k=0}^{2} \bar{m}_k(h)\, I_{k-3} \\ &= \frac{(h+1)}{\sqrt{-h}\,(\sqrt{-h}+1)}\,[P_3(h) + P_2(h)\sqrt{-h}] \\ &= \frac{1}{h}\,(hP_3(h) + \tilde{P}_3(h)\sqrt{-h}), \qquad \text{for} \quad n = 4. \end{aligned} \tag{17}$$

For $n \geqslant 5$, from (8) and (9) we obtain

$$\begin{aligned} \sum_{k=3}^{[\frac{n+1}{2}]} \bar{m}_k(h)\, I_{k-3} &= 2\sqrt{-h}\,(v_2 - v_1)^2 \sum_{k=3}^{[\frac{n+1}{2}]} h^{6-2k} P_{2k-6}(h)\, \bar{m}_k(h) \\ &= \sqrt{-h}\,(h+1)\, h^{4-2[\frac{n+1}{2}]} P_{3[\frac{n+1}{2}]-5}(h). \end{aligned} \tag{18}$$

From (8) and (10), it follows that

$$\sum_{k=3}^{[\frac{n}{2}]} m_k(h)\, J_{k-3} = 2\sqrt{-h}\,(v_2-v_1)^2 \sum_{k=3}^{[\frac{n}{2}]} h^{5-2k} P_{2k-5}(h)\, m_k(h)$$
$$= \sqrt{-h}\,(h+1)\, h^{3-2[\frac{n}{2}]} P_{3[\frac{n}{2}]-4}(h). \tag{19}$$

From (17), (18), and (19), we obtain

$$M(h) = h^{3-n}[P_3(h)\, h^{n-3} + \sqrt{-h}\, P_{\frac{3}{2}n-\frac{5}{2}}(h)], \qquad \text{for} \quad n \geqslant 5 \quad \text{odd}, \tag{20}$$
$$M(h) = h^{3-n}[P_3(h)\, h^{n-3} + \sqrt{-h}\, P_{\frac{3}{2}n-4}(h)], \qquad \text{for} \quad n \geqslant 6 \quad \text{even}. \tag{21}$$

From (15),

$$\#\{-1 < h < 0 \mid M(h) = 0\} \leqslant 2, \qquad \text{for} \quad n \leqslant 2.$$

From (16),

$$\#\{-1 < h < 0 \mid M(h) = 0\} \leqslant 5, \qquad \text{for} \quad n = 3.$$

From (17),

$$\#\{-1 < h < 0 \mid M(h) = 0\} \leqslant 6, \qquad \text{for} \quad n = 4.$$

From (20),

$$\#\{h < 0 \mid M(h) = 0\} \leqslant \tfrac{3}{2}n - \tfrac{5}{2} + 1 + 3 + 1 - 1 = \tfrac{3}{2}n + \tfrac{3}{2}.$$

On the other hand, $M(-1) = 0$; therefore,

$$\#\{-1 < h < 0 \mid M(h) = 0\} \leqslant \tfrac{3}{2}n + \tfrac{1}{2}, \qquad \text{for} \quad n \geqslant 5 \quad \text{odd}.$$

From (21)

$$\#\{h < 0 \mid M(h) = 0\} \leqslant \tfrac{3}{2}n - 4 + 1 + 3 + 1 - 1 = \tfrac{3}{2}n.$$

On the other hand, $M(-1) = 0$; therefore

$$\#\{-1 < h < 0 \mid M(h) = 0\} \leqslant \tfrac{3}{2}n - 1, \qquad \text{for} \quad n \geqslant 6 \quad \text{even}.$$

The proof of the conclusion of Theorem 2 for system A is completed.

4. LINEAR ESTIMATION OF THE NUMBER OF ZEROS OF ABELIAN INTEGRAL FOR SYSTEM B

The system B has two parallel invariant lines: $x=a\pm\sqrt{a^2+1}$, if $b=0$, $a\neq 0$, and $y=b\pm\sqrt{b^2-1}$, if $a=0$, $|b|>1$. By taking a suitable rotation and a rescaling of coordinates (if necessary) such that in the new coordinates these two invariant lines are $x=\pm 1$, system B can be written into the form

$$\dot{x}=-2y(x^2-1), \qquad \dot{y}=-x^2-2dx-1-2xy^2, \qquad d>1.$$

Consider the following polynomial perturbation of the above system,

$$\begin{aligned}\dot{x}&=-2y(x^2-1)+\varepsilon P(x,y),\\ \dot{y}&=-x^2-2dx-1-2xy^2+\varepsilon Q(x,y), \qquad d>1,\end{aligned} \tag{22}$$

where P, Q are polynomials of degree n. For $\varepsilon=0$, system (22) has a first integral $H=(y^2+x+d)/(x^2-1)$ such that $H=d_\pm=(-d\pm\sqrt{d^2-1})/2$ correspond to two centers. There are two families of periodic orbits Γ_h: $H=h$, $h\in(d_+,0)\cup(-\infty,d_-)$, which surround the centers Γ_{d_+} and Γ_{d_-}, respectively. Multiplying by the integrating factor $(x^2-1)^{-2}$, system (22) becomes

$$\begin{aligned}\dot{x}&=-\frac{\partial H}{\partial y}+\varepsilon\frac{P}{(x^2-1)^2},\\ \dot{y}&=\frac{\partial H}{\partial x}+\varepsilon\frac{Q}{(x^2-1)^2}.\end{aligned} \tag{23}$$

The Abelian integral of system (23) is defined as

$$M(h)=\oint_{H=h}\left(\frac{P}{(x^2-1)^2}\,dy-\frac{Q}{(x^2-1)^2}\,dx\right), \qquad h\in[d_+,0)\cup(-\infty,d_-].$$

Denote by D_h the simple connected region with boundary Γ_h. By Green's formula,

$$\begin{aligned}M(h)&=\iint_{D_h}\left[\frac{\partial}{\partial x}\left(\frac{P}{(x^2-1)^2}\right)+\frac{\partial}{\partial y}\left(\frac{Q}{(x^2-1)^2}\right)\right]dx\,dy\\ &=\iint_{D_h}\left[\frac{P'_x+Q'_y}{(x^2-1)^2}-\frac{4xP}{(x^2-1)^3}\right]dx\,dy.\end{aligned}$$

If

$$P'_x + Q'_y = \sum_{i+j \leqslant n-1} a_{ij} x^i y^j,$$

$$P = \sum_{i+j \leqslant n} b_{ij} x^i y^j,$$

and x_1, x_2 denote the two roots of the equation $hx^2 - x - h - d = 0$, then

$$x_1 x_2 = -1 - \frac{d}{h}, \qquad x_1 + x_2 = \frac{1}{h}, \tag{24}$$

and

$$\begin{aligned} M(h) = & \sum_{i+2j \leqslant n-1} \int_{x_1}^{x_2} \frac{2a_{i\,2j} x^i}{(2j+1)(x^2-1)^2} (hx^2 - x - h - d)^j \sqrt{hx^2 - x - h - d}\, dx \\ & - \sum_{i+2j \leqslant n} \int_{x_1}^{x_2} \frac{8b_{i\,2j} x^i}{(2j+1)(x^2-1)^3} (hx^2 - x - h - d)^j \sqrt{hx^2 - x - h - d}\, dx \\ = & \sum_{k=0}^{[\frac{n}{2}]} m_k(h)\, J_{k-3} + \sum_{k=0}^{[\frac{n+1}{2}]} \bar{m}_k(h)\, I_{k-3}, \end{aligned}$$

where m_k, $\bar{m}_k$ are polynomials of h with

$$\deg m_k \leqslant k, \qquad \deg \bar{m}_k \leqslant k, \tag{25}$$

$$I_k = \int_{x_1}^{x_2} (x^2 - 1) \sqrt{hx^2 - x - h - d}\, dx,$$

$$J_k = \int_{x_1}^{x_2} x(x^2-1)^k \sqrt{hx^2 - x - h - d}\, dx.$$

If $\sqrt{(x_2 - x)(x - x_1)} = t(x - x_1)$, then

$$I_k = 2\sqrt{-h}\,(x_2 - x_1)^2\, U_k,$$

$$U_k = \int_0^\infty \frac{t^2[(x_2 + x_1 t^2)^2 - (1+t^2)^2]^k}{(1+t^2)^{2k+3}}\, dt, \tag{26}$$

$$J_k = 2\sqrt{-h}\,(x_2 - x_1)^2\, V_k,$$

$$V_k = \int_0^\infty \frac{t^2(x_2 + x_1 t^2)[(x_2 + x_1 t^2)^2 - (1+t^2)^2]^k}{(1+t^2)^{2k+4}}\, dt. \tag{27}$$

Changing the variable t by $1/u$ in the integrals (26) and (27), we obtain

$$U_k(x_1, x_2) = U_k(x_2, x_1), \qquad V_k(x_1, x_2) = V_k(x_2, x_1).$$

Therefore for $k \geqslant 0$, U_k and V_k are symmetrical polynomials of (x_1, x_2) of degree $2k$ and $2k+1$, respectively:

$$U_k = \sum_{i+2j \leqslant 2k} c_{ij}(x_1+x_2)^i (x_1x_2)^j, \tag{28}$$

$$V_k = \sum_{i+2j \leqslant 2k+1} d_{ij}(x_1+x_2)^i (x_1x_2)^j. \tag{29}$$

Substituting (24) into (28) and (29), we obtain

$$U_k = h^{-2k} P_{2k}(h), \tag{30}$$

$$V_k = h^{-2k-1} P_{2k+1}(h). \tag{31}$$

From (26), (27), (30), and (31), we get

$$I_k = \sqrt{-h}\, h^{-2k-2}(1+4h^2+4dh)\, P_{2k}(h), \tag{32}$$

$$J_k = \sqrt{-h}\, h^{-2k-3}(1+4h^2+4dh)\, P_{2k+1}(h). \tag{33}$$

Next we calculate I_k, J_k for $k<0$. Denote by

$$I_k^{\pm} = \int_{x_1}^{x_2} (x \pm 1)^k \sqrt{(x-x_1)(x_2-x)}\, dx.$$

Let $x \pm 1 = y$, $y_i = x_i \pm 1$; then

$$I_k^{\pm} = \int_{y_1}^{y_2} y^k \sqrt{(y-y_1)(y_2-y)}\, dy.$$

Let $\sqrt{(y-y_1)(y_2-y)} = t(y-y_1)$; we have

$$I_k^{\pm} = 2(y_2-y_1)^2 \int_0^{\infty} \frac{t^2(y_2+y_1t^2)^k}{(1+t^2)^{k+3}}\, dt.$$

Computing, we obtain

$$I_{-1}^{-} = \frac{\pi}{2}\left[\frac{1}{h} - 2 + \frac{2\sqrt{d+1}}{\sqrt{-h}}\right], \qquad h \in [d_+, 0) \cup (-\infty, d_-],$$

$$I_{-2}^{-} = \frac{\pi}{2}\left[\frac{1}{\sqrt{d+1}}(-h)^{-\frac{1}{2}} - 2 + \frac{2}{\sqrt{d+1}}(-h)^{\frac{1}{2}}\right], \qquad h \in [d_+, 0) \cup (-\infty, d_-],$$

$$I_{-3}^{-} = -\frac{\pi}{8(d+1)^{\frac{3}{2}}}(-h)^{\frac{3}{2}}\left(\frac{1}{h^2} + 4 + \frac{4d}{h}\right), \qquad h \in [d_+, 0) \cup (-\infty, d_-],$$

$$I^{+}_{-1}=\begin{cases}\dfrac{\pi}{2}\left(\dfrac{1}{h}+2+\dfrac{2\sqrt{d-1}}{\sqrt{-h}}\right), & h\in[d_+,0),\\[2mm] \dfrac{\pi}{2}\left(\dfrac{1}{h}+2-\dfrac{2\sqrt{d-1}}{\sqrt{-h}}\right), & h\in(-\infty,d_-],\end{cases}$$

$$I^{+}_{-2}=\begin{cases}\dfrac{\pi}{2}\left(\dfrac{1}{\sqrt{d-1}}(-h)^{-\frac{1}{2}}-\dfrac{2(-h)^{\frac{1}{2}}}{\sqrt{d-1}}-2\right), & h\in[d_+,0),\\[2mm] \dfrac{\pi}{2}\left(-\dfrac{1}{\sqrt{d-1}}(-h)^{-\frac{1}{2}}+\dfrac{2(-h)^{\frac{1}{2}}}{\sqrt{d-1}}-2\right), & h\in(-\infty,d_-],\end{cases}$$

$$I^{+}_{-3}=\begin{cases}\dfrac{\pi}{8(d-1)^{\frac{3}{2}}}\left(-(-h)^{-\frac{1}{2}}-4(-h)^{\frac{3}{2}}+4d\sqrt{-h}\right), & h\in[d_+,0),\\[2mm] \dfrac{\pi}{8(d-1)^{\frac{3}{2}}}\left((-h)^{-\frac{1}{2}}+4(-h)^{\frac{3}{2}}-4d\sqrt{-h}\right), & h\in(-\infty,d_-].\end{cases}$$

Then

$$\begin{aligned}I_{-1}&=\int_{x_1}^{x_2}\frac{1}{x^2-1}\sqrt{hx^2-x-h-d}\,dx\\ &=\sqrt{-h}\int_{x_1}^{x_2}\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)\sqrt{(x-x_1)(x_2-x)}\,dx\\ &=\frac{\sqrt{-h}}{2}(I^{-}_{1}-I^{+}_{1})\\ &=\begin{cases}\dfrac{\pi}{2}(\sqrt{d+1}-\sqrt{d-1}-2\sqrt{-h}), & h\in[d_+,0),\\[2mm] \dfrac{\pi}{2}(\sqrt{d+1}+\sqrt{d-1}-2\sqrt{-h}), & h\in(-\infty,d_-].\end{cases}\end{aligned}$$

$$\begin{aligned}I_{-2}&=\sqrt{-h}\int_{x_1}^{x_2}\frac{1}{(x^2-1)^2}\sqrt{(x-x_1)(x_2-x)}\,dx\\ &=\frac{\sqrt{-h}}{4}\int_{x_1}^{x_2}\left[\frac{1}{(x-1)^2}+\frac{1}{(x+1)^2}-\frac{2}{x^2-1}\right]\sqrt{(x-x_1)(x_2-x)}\,dx\\ &=\frac{\sqrt{-h}}{4}(I^{-}_{-2}+I^{+}_{-2})-\frac{1}{2}I_{-1}\\ &=\begin{cases}\dfrac{\pi}{8}\left[\dfrac{1}{\sqrt{d+1}}+\dfrac{1}{\sqrt{d-1}}+2\sqrt{d-1}-2\sqrt{d+1}+\left(\dfrac{2}{\sqrt{d-1}}-\dfrac{2}{\sqrt{d+1}}\right)h\right],\\ \qquad h\in[d_+,0),\\[2mm] \dfrac{\pi}{8}\left[\dfrac{1}{\sqrt{d+1}}-\dfrac{1}{\sqrt{d-1}}-2\sqrt{d+1}-2\sqrt{d-1}-2\left(\dfrac{1}{\sqrt{d+1}}+\dfrac{1}{\sqrt{d-1}}\right)h\right],\\ \qquad h\in(-\infty,d_-].\end{cases}\end{aligned}$$

$$
\begin{aligned}
I_{-3} &= \sqrt{-h}\int_{x_1}^{x_2} \frac{1}{(x^2-1)^3}\sqrt{(x-x_1)(x_2-x)}\,dx \\
&= \frac{\sqrt{-h}}{8}\,(I_{-3}^{-} - I_{-3}^{+}) - \frac{3}{4}\,I_{-2} \\
&= \begin{cases} \dfrac{\pi}{64}\,[(d-1)^{-\frac{3}{2}} - (d+1)^{-\frac{3}{2}}](1+4dh+4h^2) - \dfrac{3}{4}\,I_{-2} & h\in[d_+,0), \\ -\dfrac{\pi}{64}\,[(d-1)^{-\frac{3}{2}} + (d+1)^{-\frac{3}{2}}](1+4dh+4h^2) - \dfrac{3}{4}\,I_{-2} & h\in(-\infty,d_-]. \end{cases} \\
J_{-1} &= \sqrt{-h}\int_{x_1}^{x_2} \frac{x}{x^2-1}\sqrt{(x-x_1)(x_2-x)}\,dx \\
&= \frac{\sqrt{-h}}{2}\,(I_{-1}^{-} + I_{-1}^{+}) \\
&= \begin{cases} \dfrac{\pi}{2}\,(\sqrt{d+1} + \sqrt{d-1} - (-h)^{-\frac{1}{2}}), & h\in[d_+,0), \\ \dfrac{\pi}{2}\,(\sqrt{d+1} - \sqrt{d-1} - (-h)^{-\frac{1}{2}}), & h\in(-\infty,d_-]. \end{cases} \\
J_{-2} &= \sqrt{-h}\int_{x_1}^{x_2} \frac{x}{(x^2-1)^2}\sqrt{(x-x_1)(x_2-x)}\,dx \\
&= \sqrt{-h}\int_{x_1}^{x_2}\left[\frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x+1)^2} + \frac{1}{(x^2-1)^2}\right]\sqrt{(x-x_1)(x_2-x)}\,dx \\
&= \sqrt{-h}\left(\frac{1}{4}\,I_{-1}^{-} - \frac{1}{4}\,I_{-1}^{+} - \frac{1}{2}\,I_{-2}^{+}\right) + I_{-2} \\
&= \frac{1}{2}\,I_{-1} - \frac{\sqrt{-h}}{2}\,I_{-2}^{+} + I_{-2} \\
&= \begin{cases} \dfrac{\pi}{8}\left(\dfrac{1}{\sqrt{d+1}} - \dfrac{1}{\sqrt{d-1}} - \left(\dfrac{2}{\sqrt{d-1}} + \dfrac{2}{\sqrt{d+1}}\right)h\right), & h\in[d_+,0), \\ \dfrac{\pi}{8}\left(\dfrac{1}{\sqrt{d+1}} + \dfrac{1}{\sqrt{d-1}} + \left(\dfrac{2}{\sqrt{d-1}} - \dfrac{2}{\sqrt{d+1}}\right)h\right), & h\in(-\infty,d_-]. \end{cases} \\
J_{-3} &= \sqrt{-h}\int_{x_1}^{x_2} \frac{x}{(x^2-1)^3}\sqrt{(x-x_1)(x_2-x)}\,dx \\
&= \sqrt{-h}\int_{x_1}^{x_2}\left[\frac{1}{8(x-1)^2} - \frac{3}{16(x-1)} + \frac{3}{16(x+1)}\right. \\
&\quad \left. + \frac{1}{4(x+1)^2} + \frac{1}{4(x+1)^3} + \frac{1}{(x^2-1)^3}\right]\sqrt{(x-x_1)(x_2-x)}\,dx
\end{aligned}
$$

$$= \sqrt{-h}\left[\frac{1}{8}I^-_{-2} - \frac{3}{16}I^-_{-1} + \frac{3}{16}I^+_{-1} + \frac{1}{4}I^+_{-2} + \frac{1}{4}I^+_{-3}\right] + I_{-3}$$

$$= \begin{cases} -\frac{\pi}{64}[(d-1)^{-\frac{3}{2}} + (d+1)^{-\frac{3}{2}}](1+4dh+4h^2) \\ \qquad + \frac{\pi}{32}\left(\frac{1}{\sqrt{d-1}} - \frac{1}{\sqrt{d+1}}\right) + \frac{\pi}{16}\left(\frac{1}{\sqrt{d-1}} + \frac{1}{\sqrt{d+1}}\right)h, \\ \qquad h \in [d_+, 0), \\ \frac{\pi}{64}[(d-1)^{-\frac{3}{2}} - (d+1)^{-\frac{3}{2}}](1+4dh+4h^2) \\ \qquad - \frac{\pi}{32}\left(\frac{1}{\sqrt{d+1}} + \frac{1}{\sqrt{d-1}}\right) + \frac{\pi}{16}\left(\frac{1}{\sqrt{d+1}} - \frac{1}{\sqrt{d-1}}\right)h, \\ \qquad h \in (-\infty, d_-]. \end{cases}$$

For $n \leqslant 2$,

$$M(h) = P_2(h). \tag{34}$$

For $n = 3$,

$$M(h) = P_2(h) + \bar{m}_2(h)\, I_{-1} = \sum_{i=0}^{5} c_i(-h)^{\frac{i}{2}}. \tag{35}$$

For $n = 4$,

$$M(h) = \sum_{i=0}^{5} c_i(-h)^{\frac{i}{2}} + m_2(h)\, J_{-1} = \sum_{i=-1}^{5} d_i(-h)^{\frac{i}{2}}. \tag{36}$$

For $n \geqslant 5$ odd,

$$\begin{aligned} M(h) &= \sum_{k=0}^{2} (m_k(h)\, J_{k-3} + \bar{m}_k(h)\, I_{k-3}) + \sum_{k=3}^{\frac{n-1}{2}} m_k(h)\, J_{k-3} + \sum_{k=3}^{\frac{n+1}{2}} \bar{m}_k(h)\, I_{k-3} \\ &= \sum_{i=-1}^{5} d_i(-h)^{\frac{i}{2}} + \sqrt{-h}\,(1+4dh+4h^2)\left[\sum_{k=3}^{\frac{n-1}{2}} m_k(h)\, h^{3-2k} P_{2k-5}(h)\right. \\ &\quad \left. + \sum_{k=3}^{\frac{n+1}{2}} \bar{m}_k(h)\, h^{4-2k} P_{2k-6}(h)\right] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=-1}^{5} d_i(-h)^{\frac{i}{2}} + \sqrt{-h}\,(1+4h^2+4dh)\left[\sum_{k=3}^{\frac{n-1}{2}} h^{3-2k} P_{3k-5}(h)\right.\\
&\quad \left. + \sum_{k=3}^{\frac{n+1}{2}} h^{4-2k} P_{3k-6}(h)\right]\\
&= \sum_{i=-1}^{5} d_i(-h)^{\frac{i}{2}} + \sqrt{-h}\,(1+4h^2+4dh) \sum_{i=3-n}^{\frac{n-5}{2}} c_i h^i\\
&= \sum_{i=-1}^{5} d_i(-h)^{\frac{i}{2}} + \sum_{i=3-n}^{\frac{n-1}{2}} \tilde{c}_i(-h)^{i+\frac{1}{2}}\\
&= a_0 + a_1 h + a_2 h^2 + \sum_{i=3-n}^{\frac{n-1}{2}} a_{i+\frac{1}{2}}(-h)^{i+\frac{1}{2}}.
\end{aligned}
\tag{37}
$$

For $n \geqslant 6$ even,

$$
\begin{aligned}
M(h) &= \sum_{k=0}^{2} (m_k(h)\, J_{k-3} + \bar{m}_k(h)\, I_{k-3}) + \sum_{k=3}^{\frac{n}{2}} m_k(h)\, J_{k-3} + \sum_{k=3}^{\frac{n}{2}} \bar{m}_k(h)\, I_{k-3}\\
&= \sum_{i=-1}^{5} d_i(-h)^{\frac{i}{2}} + \sqrt{-h}\,(1+4h^2+4dh) \sum_{k=3}^{\frac{n}{2}} (m_k(h)\, h^{3-2k} P_{2k-5}(h)\\
&\quad + \bar{m}_k(h)\, h^{4-2k} P_{2k-6}(h))\\
&= \sum_{i=-1}^{5} d_i(-h)^{\frac{i}{2}} + \sqrt{-h} \sum_{i=3-n}^{\frac{n}{2}} c_i h^i\\
&= a_0 + a_1 h + a_2 h^2 + \sum_{i=3-n}^{\frac{n}{2}} a_{i+\frac{1}{2}}(-h)^{i+\frac{1}{2}}.
\end{aligned}
\tag{38}
$$

Denote by

$$
N = \max\{\#\{h \in (d_+, 0) \mid M(h) = 0\}, \#\{h \in (-\infty, d\_) \mid M(h) = 0\}\}.
$$

From (34)–(38), we obtain

$$
\begin{aligned}
N &\leqslant 1, && \text{if} \quad n \leqslant 2;\\
N &\leqslant 4, && \text{if} \quad n = 3;\\
N &\leqslant 5, && \text{if} \quad n = 4;\\
N &\leqslant \tfrac{3}{2}(n-1), && \text{if} \quad n \geqslant 5 \quad \text{odd};\\
N &\leqslant \tfrac{3}{2} n - 1, && \text{if} \quad n \geqslant 6 \quad \text{even}.
\end{aligned}
$$

Therefore the proof of the conclusion of Theorem 2 for system B is completed.

5. LINEAR ESTIMATION OF THE NUMBER OF ZEROS OF ABELIAN INTEGRAL FOR SYSTEM C

Consider the polynomial perturbation of system C

$$\begin{aligned} \dot{x} &= -y + yx^2 + \varepsilon P(x, y), \\ \dot{y} &= x + xy^2 + \varepsilon Q(x, y), \end{aligned} \tag{39}$$

where $P = \frac{1}{2} \sum_{i+j \leqslant n} a_{ij} x^i y^j$, $Q = \frac{1}{2} \sum_{i+j \leqslant n} b_{ij} x^i y^j$. For $\varepsilon = 0$, system (39) has a first integral $H = (y^2+1)/(x^2-1)$ with integrating factor $2/(x^2-1)^2$. For $h < -1$, the level curves Γ_h: $H = h$ are periodic orbits surrounding the center Γ_{-1}: $(0, 0)$ (see Fig. 1). The Abelian integral for system (39) is defined as

$$M(h) = \oint_{H=h} \left(\frac{2P}{(x^2-1)^2} \, dy - \frac{2Q}{(x^2-1)^2} \, dx \right), \qquad -\infty < h \leqslant -1.$$

Denote by D_h the simple connected region with boundary Γ_h. By Green's formula, we have

$$\begin{aligned} M(h) &= 2 \iint_{D_h} \left(\frac{\partial (P/(x^2-1)^2)}{\partial x} + \frac{\partial (Q/(x^2-1)^2)}{\partial y} \right) dx\, dy \\ &= 2 \iint_{D_h} \left(\frac{P'_x + Q'_y}{(x^2-1)^2} - \frac{4xP}{(x^2-1)^3} \right) dx\, dy \\ &= \iint_{D_h} \left[\frac{1}{(x^2-1)^2} \left(\sum_{i+j \leqslant n} i a_{ij} x^{i-1} y^j + \sum_{i+j \leqslant n} j b_{ij} x^i y^{j-1} \right) \right. \\ &\qquad \left. - \frac{4}{(x^2-1)^3} \sum_{i+j \leqslant n} a_{ij} x^{i+1} y^j \right] dx\, dy \qquad \text{(by symmetry)} \\ &= \iint_{D_h} \left[\frac{1}{(x^2-1)^2} \left(\sum_{2i+2j \leqslant n-1} (2i+1)\, a_{2i+1\,2j} x^{2i} y^{2j} \right. \right. \\ &\qquad \left. + \sum_{2i+2j \leqslant n-1} (2j+1)\, b_{2i\,2j+1} x^{2i} y^{2j} \right) \\ &\qquad \left. - \frac{4}{(x^2-1)^3} \sum_{2i+2j \leqslant n-1} a_{2i+1\,2j} x^{2i+2} y^{2j} \right] dx\, dy. \end{aligned}$$

Denote by $\bar{x}=\sqrt{1+\frac{1}{h}}$ the positive root of equation $hx^2-h-1=0$; then

$$
\begin{aligned}
M(h) &= \int_{-\bar{x}}^{\bar{x}} \frac{1}{(x^2-1)^2} \sum_{2i+2j\leqslant n-1} \frac{(2i+1)}{(2j+1)}\, a_{2i+1\,2j} x^{2i}(hx^2-h-1)^j \sqrt{hx^2-h-1}\, dx \\
&\quad + \int_{-\bar{x}}^{\bar{x}} \frac{1}{(x^2-1)^2} \sum_{2i+2j\leqslant n-1} b_{2i\,2j+1} x^{2i}(hx^2-h-1)^j \sqrt{hx^2-h-1}\, dx \\
&\quad - \int_{-\bar{x}}^{\bar{x}} \frac{4}{(x^2-1)^3} \sum_{2i+2j\leqslant n-1} \frac{a_{2i+1\,2j}}{2j+1}\, x^{2i+2}(hx^2-h-1)^j \sqrt{hx^2-h-1}\, dx \\
&= \int_{-\bar{x}}^{\bar{x}} \sum_{k=0}^{[\frac{n+1}{2}]} m_k(h)(x^2-1)^{k-3} \sqrt{hx^2-h-1}\, dx \\
&= \sum_{k=0}^{[\frac{n+1}{2}]} m_k(h)\, I_{k-3}(h),
\end{aligned}
\tag{40}
$$

where $m_k(h)$ is a polynomial of degree $\leqslant k$ and

$$
I_k = \int_{-\bar{x}}^{\bar{x}} (x^2-1)^k \sqrt{hx^2-h-1}\, dx.
$$

If $\sqrt{\bar{x}^2-x^2}=t(x+\bar{x})$, then

$$
I_k = 8\sqrt{-h}\,\bar{x}^2 \int_0^{\infty} \frac{t^2[\bar{x}^2(1-t^2)^2-(1+t^2)^2]^k}{(1+t^2)^{2k+3}}\, dt.
$$

Computing we obtain

$$
\begin{aligned}
I_{-1} &= -\pi(\sqrt{-h}-1), \\
I_{-2} &= -\frac{\pi}{2}(1+h), \\
I_{-3} &= \frac{\pi}{8}(1+h)(3-h).
\end{aligned}
\tag{41}
$$

For $k\geqslant 0$,

$$
\begin{aligned}
I_k &= 8\sqrt{-h}\,\bar{x}^2 \int_0^{\infty} \frac{t^2 \sum_{i=0}^{k} C_k^i(-1)^{k-i}\,(1+t^2)^{2k-2i}\,(1-t^2)^{2i}\,\bar{x}^{2i}}{(1+t^2)^{2k+3}}\, dt \\
&= 8\sqrt{-h}\,\bar{x}^2 \sum_{i=0}^{k} d_{k,i}\bar{x}^{2i} \\
&= 8\sqrt{-h}\left(1+\frac{1}{h}\right) \sum_{i=0}^{k} d_{k,i}\left(1+\frac{1}{h}\right)^i \\
&= 8\sqrt{-h}\,(1+h)\, h^{-k-1} P_k(h),
\end{aligned}
\tag{42}
$$

where

$$d_{k,i} = (-1)^{k-i}\, C_k^i \int_0^\infty \frac{t^2(1+t^2)^{2k-2i}\,(1-t^2)^{2i}}{(1+t^2)^{2k+3}}\, dt,$$

and P_k is a polynomial of degree k.

From (40) and (41), we obtain

$$M(h) = m_0(h)\, I_{-3} = (1+h)(3-h)\, P_0(h), \qquad \text{if} \quad n=0; \tag{43}$$

$$M(h) = m_0(h)\, I_{-3} + m_1(h)\, I_{-2} = (1+h)\, P_1(h), \qquad \text{if} \quad n=1,2; \tag{44}$$

$$M(h) = \sum_{k=0}^{2} m_k(h)\, I_{k-3} = (\sqrt{-h}-1)(P_2(h)+P_1(h)\sqrt{-h}), \qquad \text{if} \quad n=3,4. \tag{45}$$

From (40), (41), and (42), for $n \geqslant 5$ we have

$$\begin{aligned}
M(h) &= \sum_{k=0}^{2} m_k I_{k-3} + \sum_{k=3}^{[\frac{n+1}{2}]} m_k I_{k-3} \\
&= (\sqrt{-h}-1)(P_2(h)+P_1(h)\sqrt{-h}) \\
&\quad + \sum_{k=3}^{[\frac{n+1}{2}]} 8m_k(h)\sqrt{-h}\,(1+h)\, h^{2-k} P_{k-3}(h) \\
&= P_2(h) + \tilde{P}_2(h)\sqrt{-h} + \sqrt{-h} \sum_{k=3}^{[\frac{n+1}{2}]} h^{2-k} P_{2k-2}(h) \\
&= P_2(h) + \tilde{P}_2(h)\sqrt{-h} + h^{2-[\frac{n+1}{2}]}\sqrt{-h} \sum_{k=3}^{[\frac{n+1}{2}]} P_{[\frac{n+1}{2}]+k-2} \\
&= P_2(h) + \tilde{P}_2(h)\sqrt{-h} + h^{2-[\frac{n+1}{2}]}\sqrt{-h}\, P_{2[\frac{n+1}{2}]-2} \\
&= P_2(h) + h^{2-[\frac{n+1}{2}]}\sqrt{-h}\, P_{2[\frac{n+1}{2}]-2}.
\end{aligned} \tag{46}$$

Denote by N the number of isolated zeros of $M(h)$ in $(-\infty, -1)$; then the following hold: from (43),

$$N = 0, \qquad \text{if} \quad n=0;$$

from (44),

$$N \leqslant 1, \qquad \text{if} \quad n=1,2;$$

from (45),

$$N \leqslant 4, \qquad \text{if} \quad n=3,4;$$

from (46),

$$N \leqslant 2\left[\frac{n+1}{2}\right], \qquad \text{if} \quad n \geqslant 5.$$

Hence, the proof of the conclusion for system C of Theorem 2 is completed.

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